

FUNDAMENTAL GROUP AND PLURIDIFFERENTIALS ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT. A compact Kähler manifold X is shown to be simply-connected if its ‘symmetric cotangent algebra’ is trivial. Conjecturally, such a manifold should even be rationally connected. The relative version is also shown: a proper surjective connected holomorphic map $f : X \rightarrow S$ between connected manifolds induces an isomorphism of fundamental groups if its smooth fibres are as above, and if X is Kähler.

1. INTRODUCTION

We shall show:

Theorem 1.1. *Let X be a connected compact Kähler manifold. Suppose that for all $p \geq 1$ and $k \geq 1$ there is no non-zero global section of the sheaf $S^k \Omega_X^p$. Then X is simply connected¹.*

This theorem refines a former result of [5] with the very same statement, but with $\otimes^k \Omega_X^p$ in place of $S^k \Omega_X^p$. The proof of 1.1 is obtained by refining the proof of [5], which rests on L^2 -methods à la Poincaré-Atiyah-Gromov.

The ‘uniruledness conjecture’ below implies easily (see §3) that X should, in fact, be rationally connected, hence simply-connected, by [3]. Theorem 1.1 above permits to bypass this conjecture, as far as the fundamental group is concerned. It is usually quite easy to verify the vanishings of all $S^k \Omega_X^p$, while constructing sufficiently many rational curves requires the characteristic $p > 0$ methods introduced by S. Mori, no characteristic zero proof being presently known.

The weaker assumption that $H^0(X, S^k \Omega_X^1) = \{0\}$ for every $k \geq 1$ implies (see [2]) that all linear representations of the fundamental group $\pi_1(X) \rightarrow GL_n(K)$, K a field, have finite image. This raises the question of whether the condition $H^0(X, S^k \Omega_X^1) = \{0\}$ for every $k \geq 1$ might imply that $\pi_1(X)$ is finite, instead of trivial. Enriques

¹By a theorem of Kodaira, any X as above is actually projective.

surfaces (examples of general type also exist) indeed show that simply-connectedness may then fail².

In contrast to the condition $H^0(X, S^k \Omega_X^p) = \{0\}$ for every $k \geq 1$ and $p \geq 1$, the condition $H^0(X, S^k \Omega_X^1) = \{0\}$ for every $k \geq 1$ does not seem however to have an even conjectural geometric interpretation in the frame of bimeromorphic classification of compact Kähler manifolds.

The theorem 1.1 above has a relative version, shown in section §4 below:

Corollary 1. *Let $f : X \rightarrow S$ be a proper holomorphic map with connected fibres between connected complex manifolds. Assume³ that X admits a Kähler metric, and that $f_*(S^k(\Omega_{X/S}^p)) = 0$ for every $k \geq 1$ and $p \geq 1$. Then $f_* : \pi_1(X) \rightarrow \pi_1(S)$ is an isomorphism of groups.*

Note that the conclusion of corollary 1 may fail for a projective morphism $f : X \rightarrow S$ with smooth fibres simply-connected, because of the possible presence of multiple fibres. Consider indeed an Enriques surface Y and its K3 universal cover $Y' \rightarrow Y = Y'/\mathbb{Z}_2$. Let $C \rightarrow \mathbb{P}^1 = C/\mathbb{Z}_2$ be the 2-sheeted cover defined by a hyperelliptic curve C . Now let $X \rightarrow S := \mathbb{P}^1$ be deduced from the first projection $X' := C \times Y' \rightarrow C$ by taking the equivariant quotient by the involution $u \times v$ acting freely on X' , u and v being the involutions on Y' and C respectively deduced from the \mathbb{Z}_2 covers above. Here $S = \mathbb{P}^1$ is simply connected although $\pi_1(X)$ is a \mathbb{Z}_2 extension of $\pi_1(C)$ and the smooth fibres of f are simply-connected.

2. PROOF OF THEOREM 1.1

As in [5], the proof goes in two steps: show first that $\pi_1(X)$ is finite (this is the main step, established below), and then show, using Serre's covering trick, that $\pi_1(X)$ is in fact trivial.

We start by establishing this second step. Let $\pi : X' \rightarrow X$ be a finite Galois étale cover of X of group G and degree d . The Euler characteristic of the structural sheaf of X

$$\chi(X, \mathcal{O}_X) := \sum_{i=0}^{\dim X} (-1)^i \cdot h^i(X, \mathcal{O}_X)$$

is equal to 1, since by Serre's duality $h^i(X, \mathcal{O}_X) = h^0(X, \Omega_X^i)$, and the latter is zero for $i \neq 0$ by hypothesis.

²Hopf surfaces X have $H^0(X, S^k \Omega_X^p) = \{0\}, \forall k > 1, p > 1$, showing that the Kähler assumption cannot be removed in 1.1, since $\pi_1(X) \cong \mathbb{Z}$.

³These hypothesis should imply that f is projective, locally above S .

Now, if $\omega \in H^0(X', \Omega_{X'}^i)$, the product of the $g^*\omega$ for $g \in G$ defines an element of $H^0(X', S^d \Omega_{X'}^i)$ invariant by the action of G . We obtain in this way a global section of $S^d \Omega_X^i$, which is non zero if ω is non zero. Thus it follows from the hypothesis that we must also have $\chi(X', \mathcal{O}_{X'}) = 1$.

From the multiplicativity of the Euler characteristic (see lemma 2.1 below), we get:

$$1 = \chi(X', \mathcal{O}_{X'}) = d \cdot \chi(X, \mathcal{O}_X),$$

and d is then necessarily equal to 1.

Lemma 2.1. *Let $X' \rightarrow X$ be a finite étale covering of degree d of compact complex analytic spaces. Then*

$$\chi(X', \mathcal{O}_{X'}) = d \cdot \chi(X, \mathcal{O}_X).$$

Proof. When X is projective, an elementary proof due to Kleiman is given in [12], exemple 1.1.30. In general, it is an easy consequence of the theorem of Riemann-Roch-Hirzebruch, which is proved in [14] for compact complex analytic spaces⁴. \square

. To complete the proof of theorem 1.1, we need to show that the fundamental group of X is finite. Equivalently, we have to show the

Theorem 2.2. *Let X be a connected compact Kähler manifold with infinite fundamental group. Then there exists $p \geq 1$ and $k \geq 1$ such that $H^0(X, S^k \Omega_X^p) \neq \{0\}$.*

Proof. Let $p : \tilde{X} \rightarrow X$ be the universal cover of X . The fundamental group $\Gamma := \pi_1(X)$ acts on \tilde{X} . The choice of a Kähler metric on X induces a complete Kähler metric on \tilde{X} . Denote by $\mathcal{H}_{(2)}^k(\tilde{X})$ the Hilbert space of L^2 -harmonic complex-valued forms of degree k on \tilde{X} . Recall that a p -form α is called harmonic if $\Delta\alpha = 0$, where $\Delta := d \circ d^* + d^* \circ d$ and $d^* := -* \circ d \circ *$. Moreover, a L^2 p -form α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$ (the metric being complete), if and only if $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$ (the metric being complete and Kähler), see [9].

The decomposition in types gives rise to a orthogonal sum

$$\mathcal{H}_{(2)}^k(\tilde{X}) = \bigoplus_{p+q=k} \mathcal{H}_{(2)}^{p,q}(\tilde{X}).$$

The space $\mathcal{H}_{(2)}^{p,0}(\tilde{X})$ consists of the L^2 -holomorphic p -forms on \tilde{X} .

⁴We shall only need the case when X is a divisor with normal crossings in a complex Kähler manifold in the proof of corollary 1.

The Hilbert spaces $\mathcal{H}_{(2)}^{p,q}(\tilde{X})$ might be infinite dimensional. Nevertheless, using the isometric action of Γ on them, one can associate to them a non-negative real number $\dim_{\Gamma}(\mathcal{H}_{(2)}^{p,q}(\tilde{X}))$ (cf. [1]). This number is zero if and only if $\mathcal{H}_{(2)}^{p,q}(\tilde{X}) = \{0\}$.

By Atiyah's L^2 -index theorem (cf. [1, 9]), we know that

$$\chi(X, \mathcal{O}_X) = \chi_{(2)}(\tilde{X}, \mathcal{O}_{\tilde{X}}) := \sum_{q=0}^{\dim X} (-1)^q \cdot \dim_{\Gamma}(\mathcal{H}_{(2)}^{0,q}(\tilde{X}))$$

Observe that there are no non-zero L^2 -holomorphic functions on \tilde{X} . Indeed, the metric being complete, any harmonic function is closed, hence locally constant. By hypothesis \tilde{X} is non-compact, and any constant L^2 function has to be zero.

Let us distinguish two cases. Suppose first that $\chi(X, \mathcal{O}_X) = 0$. Since $\dim H^0(X, \mathcal{O}_X) = 1$, Hodge symmetry shows that $H^0(X, \Omega_X^p) \neq \{0\}$, for some (odd) $p \geq 1$, and the theorem is proved in this case. If, now, $\chi(X, \mathcal{O}_X) \neq 0$, it follows from the discussion above that there exists $p \geq 1$ such that $\mathcal{H}_{(2)}^{0,p}(\tilde{X}) \neq \{0\}$. By conjugation $\mathcal{H}_{(2)}^{p,0}(\tilde{X}) \neq \{0\}$, hence we get a non-zero L^2 -holomorphic p -form for some $p \geq 1$.

The rest of the proof consists, following [9], in constructing from this L^2 section a non-zero Γ -invariant section of some $S^k \Omega_{\tilde{X}}^p$. This can be done using a construction which goes back to Poincaré, that we now describe in a general setting.

. Let M be a complex manifold and E be a holomorphic vector bundle on M . Let Γ be a countable discrete group acting on M and suppose that the action of Γ lifts to an action on E . Let h_E be a Γ -invariant continuous hermitian metric on E . Let $\Phi : \mathbb{P}(E) \rightarrow M$ denote the projective bundle of hyperplanes in E and $\mathcal{O}_E(1) \rightarrow \mathbb{P}(E)$ be the tautological line bundle endowed with the induced hermitian metric h_L . By functoriality the group Γ acts on $\mathbb{P}(E)$ and $\mathcal{O}_E(1)$, and all the maps considered above are Γ -equivariant. As $\Phi_*(\mathcal{O}_E(k)) = S^k E$ for all $k \geq 1$ (where $\mathcal{O}_E(k)$ denotes the line bundle $\mathcal{O}_E(1)^{\otimes k}$), there is a Γ -equivariant identification between the space of holomorphic sections $H^0(\mathbb{P}(E), \mathcal{O}_E(k)) = H^0(M, S^k E)$ under which L^q holomorphic sections are identified for all $q \geq 1$.

To any L^1 holomorphic section s of E we can associate a Γ -invariant section of $S^k E$ for all $k \geq 1$ (the so-called Poincaré series) as follows :

$$P_k(s)(x) := \sum_{\gamma \in \Gamma} \gamma^* s^k(\gamma \cdot x)$$

As s is L^1 , this series converges absolutely to a Γ -invariant holomorphic section of $S^k E$.

Moreover, if s is not the zero section, then $P_k(s)$ is non-zero for infinitely many $k \geq 1$. Indeed, the preceding construction shows that we need only to consider the case where E is a line bundle. The assertion is then a consequence of the following lemma.

Lemma 2.3. *(See Lemma 3.2.A from [9]) Let $\{a_i\}$ be an l^1 -sequence of complex numbers, not all zero. Then there are infinitely many $k \geq 1$ such that $\sum_i a_i^k \neq 0$.*

Now recall that in the case where $\chi(X, \mathcal{O}_X) \neq 0$, we showed the existence of a non-zero L^2 section s of Ω_X^p for some $p > 0$. If we see s as a section of the tautological line bundle $\mathcal{O}_{\Omega_X^p}(1)$ on the projectified bundle of Ω_X^p , then $s^{\otimes k}$ is a non-zero L^1 section of $\mathcal{O}_{\Omega_X^p}(1)$ for any $k \geq 2$. Applying the averaging construction just described to $s^{\otimes 2}$, we get a non-zero Γ -invariant section of some $\mathcal{O}_{\Omega_X^p}(2k)$, giving a non-zero section of $S^{2k}\Omega_X^p$, as claimed. This concludes the proof⁵. \square

Remark. *For any compact connected Kähler manifold X with infinite fundamental group, let $P(X)$ (resp. $P_{(2)}(X)$) be the set of integers p such that $H^0(X, S^k\Omega_X^p) \neq \{0\}$ for some $k > 0$ (resp. such that $H_{(2)}^0(X', S^k\Omega_{X'}^p) \neq \{0\}$ for some $k > 0$ and some infinite connected étale cover X' of X). The arguments above show that $P_{(2)}(X) \subset P(X)$. Complex tori show that this inclusion can be strict.*

3. A CRITERION FOR RATIONAL CONNECTEDNESS.

Recall the following consequence of the ‘Abundance Conjecture’

Conjecture. *(‘uniruledness’ conjecture) Let X be a connected compact Kähler manifold. Then X is uniruled (i.e. covered by rational curves) if and only if $H^0(X, K_X^{\otimes k}) = \{0\}$ for all $k > 0$.*

Consider also the following conjecture:

Conjecture. *Let X be a connected compact Kähler manifold. Then X is rationally connected (i.e. any two generic points are joined by some rational curve) if and only if $H^0(X, S^k\Omega_X^p) = 0$, for every $k > 0$ and $p > 0$ ⁶.*

⁵We thank C. Mourougane for observing that in our first version, our construction appeared to give a section of $S^k(S^2(\Omega_X^p))$, instead of $S^{2k}(\Omega_X^p)$.

⁶A weaker form, usually attributed to D. Mumford, claims the same conclusion assuming that $H^0(X, (\Omega_X^1)^{\otimes k}) = \{0\}$ for all $k > 0$.

In [6] a weaker form of Conjecture 3 is established: X is rationally connected if $H^0(X, S^k \Omega_X^p \otimes A) = 0$, for every $k > k(A)$, every $p > 0$, and some ample line bundle A on X .

For both conjectures, the “only if” part is easy. The second conjecture implies theorem 1.1 above, since rationally connected manifolds are simply connected [3].

Let us show that the first conjecture implies the second. First, a Kähler manifold X as in the second conjecture has $h^{2,0}(X) = 0$, so it is projective algebraic by Kodaira’s projectivity criterion. Now consider the so-called ‘rational quotient’ $r_X : X \dashrightarrow R$ (constructed in [4] and in [11], where it is called the ‘MRC’-fibration), which has rationally connected fibres and non-uniruled base R (by [7]). Assuming that $r := \dim(R) > 0$, we get a contradiction, since by the first conjecture there exists a non-zero $s \in H^0(R, K_R^{\otimes k})$, for some $k > 0$, which lifts to X as a non-zero section of $H^0(X, S^k \Omega_X^r)$. Thus $r = 0$ and X is rationally connected.

Remark. *For any compact connected Kähler manifold, let $r^-(X) := \max\{p \geq 0 \mid \exists k > 0, H^0(X, S^k \Omega_X^p) \neq \{0\}\}$. Let $r(X) := \dim(R)$, R as above. The preceding arguments show that $r(X) \geq r^-(X)$, and the uniruledness conjecture is equivalent to the equality: $r(X) = r^-(X)$.*

4. PROOF OF COROLLARY 1

The corollary 1 is an easy consequence of the theorem 1.1 and the following, the proof and statement of which are inspired by [10], theorem 5.2:

Theorem 4.1. *Let $f : X \rightarrow S$ be a proper holomorphic map with connected fibres between connected complex manifolds. Assume that X admits a Kähler metric and that there exists a smooth fibre X_s of f which is simply-connected and satisfies $H^p(X_s, \mathcal{O}_{X_s}) = 0$ for all $p > 0$. Then $f_* : \pi_1(X) \rightarrow \pi_1(S)$ is an isomorphism of groups.*

Proof. First observe that all the smooth fibres X_s of f are simply-connected and satisfy $H^p(X_s, \mathcal{O}_{X_s}) = 0$ for all $p > 0$. Indeed, the restriction of f to its smooth locus $S^\circ \subset S$ is topologically a locally trivial fiber bundle by Ehresmann’s lemma, and the dimension of $H^p(X_s, \mathcal{O}_{X_s})$ is locally constant for $s \in S^\circ$, as follows from the theory of variations of Hodge structures.

Let us first consider the following special case: X is a connected complex Kähler manifold, $f : X \rightarrow \Delta$ is a proper holomorphic map with connected fibres, smooth outside $0 \in \Delta$. Recall that in this situation X_0 is a retract of X . We have to show that the fundamental group of

X (which is isomorphic to $\pi_1(X_0)$) is trivial. By blowing-up X , one can ensure that X_0 has only simple normal crossings (i.e. the irreducible components of the corresponding reduced divisor are smooth and meet transversally); this does not change the fundamental group of X . By ([10], lemma 5.2.2) the fundamental group of X is finite cyclic, say of order d . Let $\pi : \tilde{X} \rightarrow X$ be a universal cover of X and $g : \tilde{X} \rightarrow \Delta$ be the Stein factorization of $f \circ \pi$ so that:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \Delta \\ \downarrow \pi & & \downarrow t \mapsto t^d \\ X & \xrightarrow{f} & \Delta \end{array}$$

The fibre \tilde{X}_t of g at any $t \neq 0$ is isomorphic to X_{t^d} , hence $H^p(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}) = H^p(X_{t^d}, \mathcal{O}_{X_{t^d}}) = 0$ for $t \neq 0$ and $p > 0$, and the sheaves $R^p g_* \mathcal{O}_{\tilde{X}}$ are generically zero for all $p > 0$. Being torsion-free (see [16], theorem 2.11⁷), they are in fact zero on Δ . Using Leray's spectral sequence, this implies that $H^p(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^p(\Delta, g_* \mathcal{O}_{\tilde{X}}) = 0$ for $p > 0$. Applying the lemma 4.2 below, it follows that $H^p(\tilde{X}_0^{red}, \mathcal{O}_{\tilde{X}_0^{red}}) = 0$ for all $p > 0$, hence $\chi(\tilde{X}_0^{red}, \mathcal{O}_{\tilde{X}_0^{red}}) = 1$. By multiplicativity of the holomorphic Euler characteristic in finite étale cover (see lemma 2.1), $d = 1$ and X is simply-connected.

Lemma 4.2. (*Steenbrink, see [17] lemma 2.14 and [10] lemma 5.2.3*)
Let X be a complex Kähler manifold and let $D \subset X$ be a reduced divisor such that D as a complex space is proper and has normal crossing only. Assume moreover that D is topologically a retract of X . Then the restriction maps $H^p(X, \mathcal{O}_X) \rightarrow H^p(D, \mathcal{O}_D)$ are surjective for all $p \geq 0$.

Proof. Fix a $p \geq 0$. Since D is topologically a retract of X , the map $H^p(X, \mathbb{C}) \rightarrow H^p(D, \mathbb{C})$ is an isomorphism. On the other hand, as D is a union of compact Kähler manifolds crossing transversally, $H^p(D, \mathbb{C})$ admits a canonical mixed Hodge structure (see [8] section 4) whose Hodge filtration $H^p(D, \mathbb{C}) = F^0 H^p(D, \mathbb{C}) \supseteq F^1 H^p(D, \mathbb{C}) \supseteq \dots$ satisfies $Gr_F^0 H^p(D, \mathbb{C}) \cong H^p(D, \mathcal{O}_D)$, see [17] section (1.5). It follows that

⁷in this reference the morphism is supposed projective but the same proof works for a proper morphism assuming that the total space admits a Kähler metric. See also [15], corollary 11.18.

the map $H^p(D, \mathbb{C}) \rightarrow H^p(D, \mathcal{O}_D)$ is surjective. The following commutative diagram

$$\begin{array}{ccc} H^p(X, \mathbb{C}) & \longrightarrow & H^p(X, \mathcal{O}_X) \\ \parallel & & \downarrow \\ H^p(D, \mathbb{C}) & \longrightarrow & H^p(D, \mathcal{O}_D) \end{array}$$

shows that $H^p(X, \mathcal{O}_X) \rightarrow H^p(D, \mathcal{O}_D)$ is surjective. \square

We now reduce the general case to this special case. First, because of the following diagram, theorem 4.1 for f follows from the corresponding statement for the restriction of f to an open $U := S - T$, if the codimension in S of T , Zariski closed in S , is at least 2:

$$\begin{array}{ccc} \pi_1(f^{-1}(U)) & \xrightarrow{f_*} & \pi_1(U) \\ \downarrow & & \parallel \\ \pi_1(X) & \xrightarrow{f_*} & \pi_1(S) \end{array}$$

On the other hand, any $s \in S$ admits a contractible neighborhood U in S such that $f^{-1}(U)$ is homeomorphic to $U \times f^{-1}(s)$ (see for example [13]). From this, one easily sees that the theorem 4.1 for $f : X \rightarrow S$ follows if all fibres X_s are simply-connected, at least for s outside a codimension ≥ 2 closed subvariety by the preceding observation.

Let $D \subset S$ be the proper closed subset of points s for which X_s is not smooth. By removing a codimension ≥ 2 subvariety of S , one can assume that D is a smooth divisor in S . Now, an easy application of Sard's lemma shows that for $s \in D$ outside a proper subvariety $Z \subset D$, there exists a small disk Δ_s crossing D transversally at s such that $f^{-1}(\Delta_s)$ is smooth. For any $s \in D - Z$, the restriction of f to Δ_s satisfies the assumptions of the special case of theorem 4.1 that we showed above, hence $\pi_1(X_s) = \pi_1(f^{-1}(\Delta_s)) = \{1\}$. \square

Let us now explain how the theorems 1.1 and 4.1 imply the corollary 1. First observe that for fixed $k > 0$ and $p > 0$, the dimension of $H^0(X_s, (S^k \Omega_{X/S}^p)_{|X_s})$ is constant on a non empty Zariski open subset of S , and this dimension has to be zero by the flat base change theorem. It follows that $H^0(X_s, S^k \Omega_{X_s}^p) = 0$ for all $k > 0$ and $p > 0$ for a general smooth fibre X_s of f . By theorem 1.1 this implies that a general smooth fibre of f is simply connected; hence every smooth fibre is simply connected. The same argument shows that, in particular, for all $p > 0$, $h^0(X_s, \Omega_{X_s}^p) = 0$, for $s \in S$ generic, and so: $h^p(X_s, \mathcal{O}_{X_s}) = 0$

by Hodge symmetry. We can thus apply theorem 4.1 to conclude the proof of corollary 1.

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